UPPSALA UNIVERSITET
Matematiska Institutionen
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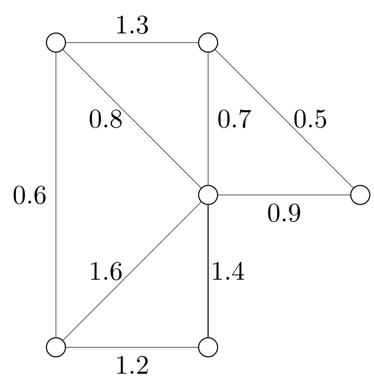
PROV I MATEMATIK
Graph Theory
10 January 2017

Time: 8.00-13.00.
Each question is worth 5 marks. You need 9 marks to pass the exam, which is one of the requirements for passing the course. You may write your answers in English or Swedish. Good luck!

1. (a) What is a planar graph?
   (b) State Euler's formula relating the number of vertices, edges and faces of a connected planar graph.
   (c) Prove Euler's formula for connected planar graphs.
   (d) Let G be a planar graph consisting of three connected components. Suppose it has 11 edges and 4 faces. How many vertices must it have?

2. (a) Let G be a weighted graph. What is meant by a minimum spanning tree of G? [You should define what is meant by the words 'minimum', 'spanning' and 'tree' in this context.]
   (b) Outline Kruskal's *or* Prim's algorithm for finding a minimum spanning tree of a weighted graph.
   (c) Follow your outline in (b) to determine a minimum spanning tree of the following weighted graph.

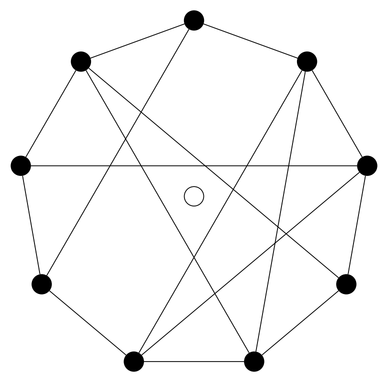


   (d) Does this weighted graph have a unique minimum spanning tree? Explain your answer.

3. (a) Explain what it means for a graph to be Eulerian.

   (b) Provide, with proof, a sufficient and necessary condition for a graph to be Eulerian.

   (c) In the graph below, you may add edges between the central white vertex and any black vertex. Is it possible to do so in such a way that the resulting graph is Eulerian?



   (d) Let $G = (V(G), E(G))$ be a connected graph with at least 2 vertices. Suppose all vertices are coloured black. Add to G a white vertex, along with some edges, each of which must connect the white vertex to some black vertex. Is it always possible to do this in such a way that the resulting graph is Eulerian? If no, provide a counter–example. If yes, specify precisely which black vertices should get new edges and show that the resulting graph is Eulerian.

4. a) How is the random graph $G(n, p(n))$ constructed?

   b) Derive a formula for the expected number of triangles in $G(n, p(n))$.

   c) In lectures we saw the following result. *If X is a non-negative integer-valued random variable, then* $\mathbb{P}[X > 0] \leq \mathbb{E}[X]$.

   Denote by a_n the probability that $G(n, p(n))$ contains at least one triangle. Show that $a_n \to 0$ as $n \to \infty$ if $p(n) = n^{-3/2}$.